Arithmetic, geometric, and harmonic means for accretive-dissipative matrices

Minghua Lin

Abstract

The concept of Loewner (partial) order for general complex matrices is introduced. After giving the definition of arithmetic, geometric, and harmonic mean for accretive-dissipative matrices, we study their basic properties. An AM-GM-HM inequality is established for two accretive-dissipative matrices in the sense of this extended Loewner order. We also compare the harmonic mean and parallel sum of two accretive-dissipative matrices, revealing an interesting relation between them. A number of examples are included.

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1 Introduction

Let $\mathbb{M}_n$ be the space of complex matrices of size $n \times n$. For any $T \in \mathbb{M}_n$, T^* means the conjugate transpose of T and we can write

$$T = A + iB, \tag{1.1}$$

in which $A = \frac{T+T^*}{2}$ and $B = \frac{T-T^*}{2i}$ are both Hermitian. This is called the Toeplitz decomposition (sometimes also called Cartesian decomposition) of T. A and B are called real part and imaginary part of T, respectively. For Hermitian matrices, there is an important partial order called Loewner (partial) order which says that for two Hermitian matrices $A, B \in \mathbb{M}_n$, $A > (\geq) B$ provided $A - B$ is positive (semi)definite. However, unlike the fruitful result of Loewner order, partial order for general complex matrices seems to be less investigated. The unique decomposition (1.1) enables us to give a natural extension of the Loewner order for general complex matrices.

Let $T, S \in \mathbb{M}_n$, with their Toeplitz decompositions

$$T = A + iB, \quad S = C + iD. \tag{1.2}$$

We define the partial order

$$T > (\geq) S$$

provided that both real and imaginary parts of T and S enjoy the same Loewner order.

A matrix $T \in \mathbb{M}_n$ is said to be *accretive-dissipative* if, in its Toeplitz decomposition (1.1), both matrices A and B are positive definite[1]. The set of accretive-dissipative matrices of order n will be denoted by $\mathbb{M}_n^{++}$. Thus $T \in \mathbb{M}_n^{++}$ is the same as $T > 0$. One checks that $\mathbb{M}_n^{++}$ forms a cone. There are several recent works devoted to studying this kind of matrix (see [8, 11]) and more generally, matrices with positive real part (see [2, 3, 16]).

Our main consideration is the possible extension of the classical Loewner order to the generalized Loewner order for general complex matrices, especially for accretive-dissipative matrices. The latter behaves differently from the former. Here is a quick example:

Let $T \in \mathbb{M}_n^{++}$, then (see [12, p. 281]) there is a unique square root of T, denoted by $T^{\frac{1}{2}}$, that belongs to $\mathbb{M}_n^{++}$. The famous Loewner theorem (see [1]) states that

For Hermitian positive definite matrices $A, C \in \mathbb{M}_n$,

$$\textit{if } A \geq C \quad \textit{ensures} \quad A^r \geq C^r \quad \textit{for any} \quad r \in [0, 1].$$

But this fails for accretive-dissipative matrices as the following explicit example shows,

Example 1.1. *Let $T = 32I + 24iI$, $S = 7I + 24iI$ (throughout, I denotes the identity matrix of an appropriate size). Obviously, $T \geq S$. However, $T^{\frac{1}{2}} = 6I + 2iI$, $S^{\frac{1}{2}} = 4I + 3iI$, so we don't have $T^{\frac{1}{2}} \geq S^{\frac{1}{2}}$.*

In Section 2, we define the arithmetic, geometric, and harmonic means for accretive-dissipative matrices and study their basic properties. We present an AM-GM-HM inequality in the sense of this generalized Loewner order. In Section 3, a comparison between harmonic mean and the parallel sum of two accretive-dissipative matrices reveals an interesting relation between them. In the final section, we present some discussion related to partial orders involving Schur complements and make some concluding remarks.

2 Arithmetic, Geometric and Harmonic means

For two Hermitian positive definite matrices $A, C \in \mathbb{M}_n$, we use the following notation for the arithmetic, geometric and harmonic means of A and C, respectively (see [13]):

$$\begin{aligned} A \nabla C &= \frac{A + C}{2}, \\ A \sharp C &= A^{\frac{1}{2}} (A^{-\frac{1}{2}} C A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}, \\ A ! C &= 2(A^{-1} + C^{-1})^{-1}. \end{aligned}$$

In this section, we extend these three means to the cone of accretive-dissipative matrices. For $T, S \in \mathbb{M}_n^{++}$, the arithmetic mean of them is naturally defined (using the

[1] Since the numerical range and hence the eigenvalues of T are in the first quadrant, another terminology for this kind of matrix is *first quadrant matrix*.

same notation) by

$$T\nabla S = \frac{T+S}{2} = A\nabla C + iB\nabla D.$$

Our definitions of geometric mean and harmonic mean are not too surprising. They are defined, respectively, as

$$T\sharp S = A\sharp C + iB\sharp D;$$

$$T!S = A!C + iB!D.$$

One finds that $T\sharp S = S\sharp T$. As we shall see, these definition are natural enough to derive many analogue properties in the usual sense.

The first one is an analogue of the maximal characterization of geometric mean for positive definite matrices.

Proposition 2.1. *Let $T, S \in \mathbb{M}_n^{++}$. Then*

$$T\sharp S = \max\left\{X \;\middle|\; \begin{bmatrix} T & X \\ X & S \end{bmatrix} \geq 0\right\}. \tag{2.1}$$

The "maximum" here is in the sense of partial order.

Proof. Let $X = Y + iZ$ be the Toeplitz decomposition of X. Then

$$\begin{bmatrix} T & X \\ X & S \end{bmatrix} \geq 0$$

means

$$\begin{bmatrix} A & Y \\ Y & C \end{bmatrix} \geq 0 \quad \text{and} \quad \begin{bmatrix} B & Z \\ Z & D \end{bmatrix} \geq 0. \tag{2.2}$$

The (unique) maximums of such Hermitian matrices Y, Z such that (2.2) holds are known (see [1]) to be $A\sharp C$, $B\sharp D$, respectively. In particular, $A\sharp C + iB\sharp D \in \left\{X \;\middle|\; \begin{bmatrix} T & X \\ X & S \end{bmatrix} \geq 0\right\}$. Now we shall show $T\sharp S = A\sharp C + iB\sharp D$ is the (unique) maximum of $\left\{X \;\middle|\; \begin{bmatrix} T & X \\ X & S \end{bmatrix} \geq 0\right\}$. For any $X_0 = Y_0 + iZ_0 \in \left\{X \;\middle|\; \begin{bmatrix} T & X \\ X & S \end{bmatrix} \geq 0\right\}$, then $\begin{bmatrix} A & Y_0 \\ Y_0 & C \end{bmatrix} \geq 0$ and $\begin{bmatrix} B & Z_0 \\ Z_0 & D \end{bmatrix} \geq 0$. Thus $A\sharp C \geq Y_0$ and $B\sharp D \geq Z_0$, i.e., $T\sharp S \geq X_0$. This completes the proof. $\square$

The following corollary is an application of Proposition 2.1.

Corollary 2.2. *Let $T_k, S_k \in \mathbb{M}_n^{++}$ for $k = 1, \ldots, m$. Then*

$$\left(\sum_{k=1}^m T_k\right) \sharp \left(\sum_{k=1}^m S_k\right) \geq \sum_{k=1}^m T_k \sharp S_k.$$

Proof. By Proposition 2.1, we have $\begin{bmatrix} T_k & T_k\sharp S_k \\ T_k\sharp S_k & S_k \end{bmatrix} \geq 0$ for $k = 1,\ldots,m$. Thus $\begin{bmatrix} \sum_{k=1}^m T_k & \sum_{k=1}^m T_k\sharp S_k \\ \sum_{k=1}^m T_k\sharp S_k & \sum_{k=1}^m S_k \end{bmatrix} \geq 0$. By Proposition 2.1 again, we have

$$\left(\sum_{k=1}^m T_k\right)\sharp\left(\sum_{k=1}^m S_k\right) \geq \sum_{k=1}^m T_k\sharp S_k,$$

as desired. □

Proposition 2.3. *Let $T, S \in \mathbb{M}_n^{++}$, then*

$$T!S = \max\left\{X \,\middle|\, \begin{bmatrix} 2T & 0 \\ 0 & 2S \end{bmatrix} \geq \begin{bmatrix} X & X \\ X & X \end{bmatrix}\right\}. \tag{2.3}$$

Again, the "maximum" is in the sense of partial order.

Proof. Writing $X = Y + iZ$ to be the Toeplitz decomposition of X, then

$$\begin{bmatrix} 2T & 0 \\ 0 & 2S \end{bmatrix} \geq \begin{bmatrix} X & X \\ X & X \end{bmatrix}$$

means

$$\begin{bmatrix} 2A & 0 \\ 0 & 2C \end{bmatrix} + i\begin{bmatrix} 2B & 0 \\ 0 & 2D \end{bmatrix} \geq \begin{bmatrix} Y & Y \\ Y & Y \end{bmatrix} + i\begin{bmatrix} Z & Z \\ Z & Z \end{bmatrix}.$$

Thus the maximum of X, say $X_{\max}$, in $\left\{X \,\middle|\, \begin{bmatrix} 2T & 0 \\ 0 & 2S \end{bmatrix} \geq \begin{bmatrix} X & X \\ X & X \end{bmatrix}\right\}$ is the same as the (unique) maximums of Hermitian matrices Y and Z such that

$$\begin{bmatrix} 2A & 0 \\ 0 & 2C \end{bmatrix} \geq \begin{bmatrix} Y & Y \\ Y & Y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2B & 0 \\ 0 & 2D \end{bmatrix} \geq \begin{bmatrix} Z & Z \\ Z & Z \end{bmatrix}.$$

The maximums of such Hermitian matrices Y and Z are known (see [1]) to be $A!C$, $B!D$, respectively. Thus,

$$X_{\max} = A!C + iB!D = T!S,$$

as desired. □

In the same vein, we have the following corollary, whose proof is left to the interested reader.

Corollary 2.4. *Let $T_k, S_k \in \mathbb{M}_n^{++}$ for $k = 1,\ldots,m$. Then*

$$\left(\sum_{k=1}^m T_k\right)!\left(\sum_{k=1}^m S_k\right) \geq \sum_{k=1}^m T_k!S_k.$$

With the notion of extended Loewner order, we can state our AM-GM-HM inequality for accretive-dissipative matrices.

Theorem 2.5. *Let $T, S \in \mathbb{M}_n^{++}$, then*

$$T \nabla S \geq T \sharp S \geq T ! S. \tag{2.4}$$

As the geometric mean for two positive definite matrices is quite intriguing, we shall explore more on this aspect for accretive-dissipative matrices.

Proposition 2.6. *Let $T, S \in \mathbb{M}_n^{++}$, then for any nonsingular $Q \in \mathbb{M}_n$, we have*

$$(Q^*TQ) \sharp (Q^*SQ) = Q^*(T \sharp S)Q.$$

Proof. We write T, S as in (1.2), then

$$\begin{aligned} (Q^*TQ) \sharp (Q^*SQ) &= (Q^*AQ + iQ^*BQ) \sharp (Q^*CQ + iQ^*DQ) \\ &= (Q^*AQ) \sharp (Q^*CQ) + i(Q^*BQ) \sharp (Q^*DQ) \\ &= Q^*(A \sharp C)Q + iQ^*(B \sharp D)Q \\ &= Q^*(A \sharp C + iB \sharp D)Q \\ &= Q^*(T \sharp S)Q, \end{aligned}$$

where the third equality is by the well known property of geometric mean for positive definite matrices. $\square$

Proposition 2.7. *Let $T, S \in \mathbb{M}_n^{++}$. If $TS = ST$ and either S or T is normal, then*

$$T \sharp S = A^{\frac{1}{2}} C^{\frac{1}{2}} + iB^{\frac{1}{2}} D^{\frac{1}{2}}.$$

Proof. We may assume that S is normal and write T, S as in (1.2). Since $TS = ST$, then by Fuglede's theorem [7], we have $TS^* = S^*T$ and so $ST^* = T^*S$. Hence

$$(T + T^*)(S + S^*) = (S + S^*)(T + T^*)$$

$$(T - T^*)(S - S^*) = (S - S^*)(T - T^*)$$

i.e.,

$$AC = CA, \quad BD = DB.$$

Therefore, $T \sharp S = A \sharp C + iB \sharp D = A^{\frac{1}{2}} C^{\frac{1}{2}} + iB^{\frac{1}{2}} D^{\frac{1}{2}}$. $\square$

The following example shows that generally we don't have $T \sharp S = T^{\frac{1}{2}} S^{\frac{1}{2}}$ even when both T, S are normal and commute.

Example 2.8. *Let $T = 3I + 4iI$, $S = 15I + 8iI$, then $T^{\frac{1}{2}} = 2I + iI$, $S^{\frac{1}{2}} = 4I + iI$. Obviously, T, S are normal and commute. However,*

$$T \sharp S = 3\sqrt{5}I + 4\sqrt{2}iI \neq 7I + 6iI = T^{\frac{1}{2}} S^{\frac{1}{2}}.$$

Proposition 2.9. *Let $T, S \in \mathbb{M}_n^{++}$, then*

$$T \sharp S \leq S \Leftrightarrow T \leq S \Leftrightarrow T \leq T \sharp S.$$

Proof. This follows immediately from the fact (e.g. [5, Theorem 4.2]) that for positive definite matrices $A, C \in \mathbb{M}_n$,

$$A\sharp C \leq C \Leftrightarrow A \leq C \Leftrightarrow A \leq A\sharp C.$$

$\square$

It is well known that the geometric mean of two positive definite matrices $A, C \in \mathbb{M}_n$ is a solution of the Riccati equation $XA^{-1}X = C$. However, this fails for accretive-dissipative matrices.

Example 2.10. *Let $T = I + iI$, $S = I + 2iI$, then $T\sharp S = I + \sqrt{2}iI$. It is easy to see that*

$$(T\sharp S)T^{-1}(T\sharp S) \neq S.$$

3 Harmonic mean and Parallel sum

Let $T \in \mathbb{M}_n^{++}$ as in (1.1), set

$$T^{-1} = E + iF, \quad E = E^*, F = F^*.$$

Then it is known [16],

$$E = (A + BA^{-1}B)^{-1}, \quad F = -(B + AB^{-1}A)^{-1},$$

from which it is clear that the imaginary part of T^{-1} is negative definite.

In a similar manner, we can show,

$$\begin{aligned} T = (T^{-1})^{-1} &= (E + iF)^{-1} \\ &= (E + FE^{-1}F)^{-1} - i(F + EF^{-1}E)^{-1}. \end{aligned}$$

Thus, comparing the real and imaginary part, we have the following identities:

$$\begin{aligned} A^{-1} &= (A + BA^{-1}B)^{-1} \\ &\quad +(B + AB^{-1}A)^{-1}(A + BA^{-1}B)(B + AB^{-1}A)^{-1}, \quad (3.1) \\ B^{-1} &= (B + AB^{-1}A)^{-1} \\ &\quad +(A + BA^{-1}B)^{-1}(B + AB^{-1}A)(A + BA^{-1}B)^{-1}. \quad (3.2) \end{aligned}$$

Using Sherman-Morrison-Woodbury matrix identity (e.g. [9]), we have

$$(A + BA^{-1}B)^{-1} = A^{-1} - A^{-1}B(A + BA^{-1}B)^{-1}BA^{-1},$$

after some rearrangement, one could indeed verify the above two identities.

Recall that for any nonsingular matrices $X, Y \in \mathbb{M}_n$ such that $X + Y$ is also nonsingular, the parallel sum $X : Y$ is given by

$$X : Y = (X^{-1} + Y^{-1})^{-1}.$$

Thus, $A!C = 2(A : C)$ for Hermitian positive definite matrices $A, C \in \mathbb{M}_n$.

Using a property of accretive-dissipative matrices (see [8, Property 1]), we know that $T : S \in \mathbb{M}_n^{++}$, provided both $T, S \in \mathbb{M}_n^{++}$. It is curious to know the relation between $T!S$ and $T : S$ for $T, S \in \mathbb{M}_n^{++}$. The main result of this section is devoted to this problem.

Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{M}_n \tag{3.3}$$

be comformably partitioned such that the diagonal blocks are square. If A_{22} is nonsingular, then the Schur complement of A_{22} in A is denoted by $A/A_{22} := A_{11} - A_{12}A_{22}^{-1}A_{21}$.

Lemma 3.1. *Consider the block matrices A, B partitioned as in (3.3). If A_{22}, B_{22} and $A_{22} + B_{22}$ are nonsingular, then*

$$(A + B)/(A_{22} + B_{22}) = A/A_{22} + B/B_{22} + X(A_{22} : B_{22})Y, \tag{3.4}$$

where $X = A_{12}A_{22}^{-1} - B_{12}B_{22}^{-1}$ and $Y = A_{22}^{-1}A_{21} - B_{22}^{-1}B_{21}$.

Proof. The condition ensures that $A_{22} : B_{22} = (A_{22}^{-1} + B_{22}^{-1})^{-1}$ is well defined.

$$\begin{aligned}
& X(A_{22}^{-1} + B_{22}^{-1})^{-1}Y \\
= \;& (A_{12}A_{22}^{-1} - B_{12}B_{22}^{-1})B_{22}(A_{22} + B_{22})^{-1}A_{22}(A_{22}^{-1}A_{21} - B_{22}^{-1}B_{21}) \\
= \;& (A_{12}A_{22}^{-1}B_{22} - B_{12})(A_{22} + B_{22})^{-1}(A_{21} - A_{22}B_{22}^{-1}B_{21}) \\
= \;& (A_{12}A_{22}^{-1}B_{22} + A_{12} - A_{12} - B_{12})(A_{22} + B_{22})^{-1}(A_{21} - A_{22}B_{22}^{-1}B_{21}) \\
= \;& (A_{12}A_{22}^{-1}B_{22} + A_{12})(A_{22} + B_{22})^{-1}(A_{21} - A_{22}B_{22}^{-1}B_{21}) \\
& -(A_{12} + B_{12})(A_{22} + B_{22})^{-1}(A_{21} - A_{22}B_{22}^{-1}B_{21}) \\
= \;& A_{12}A_{22}^{-1}(A_{21} - A_{22}B_{22}^{-1}B_{21}) \\
& -(A_{12} + B_{12})(A_{22} + B_{22})^{-1}(A_{21} + B_{21} - B_{21} - A_{22}B_{22}^{-1}B_{21}) \\
= \;& A_{12}A_{22}^{-1}A_{21} - A_{12}B_{22}^{-1}B_{21} - (A_{12} + B_{12})(A_{22} + B_{22})^{-1}(A_{21} + B_{21}) \\
& +(A_{12} + B_{12})(A_{22} + B_{22})^{-1}(B_{21} + A_{22}B_{22}^{-1}B_{21}) \\
= \;& A_{12}A_{22}^{-1}A_{21} - A_{12}B_{22}^{-1}B_{21} - (A_{12} + B_{12})(A_{22} + B_{22})^{-1}(A_{21} + B_{21}) \\
& +(A_{12} + B_{12})B_{22}^{-1}B_{21} \\
= \;& A_{12}A_{22}^{-1}A_{21} + B_{12}B_{22}^{-1}B_{21} - (A_{12} + B_{12})(A_{22} + B_{22})^{-1}(A_{21} + B_{21}).
\end{aligned}$$

(3.4) becomes clear after writing down $(A+B)/(A_{22}+B_{22})$, A/A_{22} and B/B_{22} explicitly. □

Remark 3.2. The formula (3.4) presents a relation between the Schur complement of a sum and the sum of Schur complements. A related formula for the difference of two Schur complements can be found in [4].

The special case where A, B are Hermitian in Lemma 3.1 is of particular interest. In this case, $Y = X^*$. The corollary below is an immediate consequence of Lemma 3.1.

Corollary 3.3. *[6, Theorem 1] Let $A, B \in \mathbb{M}_n$ be positive definite and partitioned as in (3.3), then*

$$(A+B)/(A_{22}+B_{22}) \geq A/A_{22} + B/B_{22}. \tag{3.5}$$

Proof. In this case, one observes that $X(A_{22}^{-1}+B_{22}^{-1})^{-1}X^* \geq 0$, where $X = A_{12}A_{22}^{-1} - B_{12}B_{22}^{-1}$. □

In relating to (3.5), we have shown in [15, Lemma 2.2] the following proposition:

Proposition 3.4. *Let $A, B \in \mathbb{M}_n$ be positive definite and partitioned as in (3.3), then*

$$(A+iB)/(A_{22}+iB_{22}) \geq A/A_{22} + iB/B_{22}. \tag{3.6}$$

Now we are going to present the main result of this section.

Theorem 3.5. *Let $T, S \in \mathbb{M}_n^{++}$ as in (1.2), then*

$$2(T:S) \geq T!S.$$

Proof. Denoted by $M = (A+BA^{-1}B)^{-1} + (C+DC^{-1}D)^{-1}$, $N = (B+AB^{-1}A)^{-1} + (D+CD^{-1}C)^{-1}$, then

$$T:S = (M+NM^{-1}N)^{-1} + i(N+MN^{-1}M)^{-1}.$$

From the expression

$$\begin{aligned} \begin{bmatrix} M & N \\ N & -M \end{bmatrix} &= \begin{bmatrix} (A+BA^{-1}B)^{-1} & (B+AB^{-1}A)^{-1} \\ (B+AB^{-1}A)^{-1} & -(A+BA^{-1}B)^{-1} \end{bmatrix} \\ &\quad + \begin{bmatrix} (C+DC^{-1}D)^{-1} & (D+CD^{-1}C)^{-1} \\ (D+CD^{-1}C)^{-1} & -(C+DC^{-1}D)^{-1} \end{bmatrix} \end{aligned}$$

and by Lemma 3.1, we have

$$\begin{aligned} M+NM^{-1}N &= (A+BA^{-1}B)^{-1} + (B+AB^{-1}A)^{-1}(A+BA^{-1}B)(B+AB^{-1}A)^{-1} \\ &\quad +(C+DC^{-1}D)^{-1} + (D+CD^{-1}C)^{-1}(C+DC^{-1}D)(D+CD^{-1}C)^{-1} \\ &\quad +\text{a Hermitian negative definite matrix, say } R \\ &= A^{-1}+C^{-1}+R \\ &\leq A^{-1}+C^{-1}, \end{aligned}$$

where the second equality is by (3.1) and (3.2).

Thus $(A^{-1}+C^{-1})^{-1} \leq (M+NM^{-1}N)^{-1}$. The role of A, C, M and B, D, N are symmetric, so we also have $(B^{-1}+D^{-1})^{-1} \leq (N+MN^{-1}M)^{-1}$. This completes the proof. □

The following example shows that there is not a partial order for $2(T:S)$ and $T\sharp S$ for $T, S \in \mathbb{M}_n^{++}$.

Example 3.6. *Let $T = I + iI$, $S = I + 2iI$, then*

$$T\sharp S = I + \sqrt{2}iI, \quad 2(T:S) = \frac{14}{13}I + \frac{6}{13}iI.$$

There is no ordering between $2(T:S)$ and $T\sharp S$ in this case.

4 Concluding remarks

4.1 Discussions

For positive definite matrices, the following proposition is known (see [14]).

Proposition 4.1. *Let $A, C \in \mathbb{M}_n$ be positive definite and for any binary operation $\sigma \in \{\nabla, \sharp, !\}$, we partition A, C and $A\sigma C$ comformably as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad A\sigma C = \begin{bmatrix} (A\sigma C)_{11} & (A\sigma C)_{12} \\ (A\sigma C)_{21} & (A\sigma C)_{22} \end{bmatrix}$$

such that the diagonal blocks are square. Then

$$(A\sigma C)/(A\sigma C)_{22} \geq (A/A_{22})\sigma(C/C_{22}).$$

It is thus natural to ask the following question.

Question 4.2. *Let $T, S \in \mathbb{M}_n^{++}$ and for any binary operation $\sigma \in \{\nabla, \sharp, !\}$, we partition T, S and $T\sigma S$ comformably as*

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad T\sigma S = \begin{bmatrix} (T\sigma S)_{11} & (T\sigma S)_{12} \\ (T\sigma S)_{21} & (T\sigma S)_{22} \end{bmatrix}$$

such that the diagonal blocks are square. Do we have

$$(T\sigma S)/(T\sigma S)_{22} \geq (T/T_{22})\sigma(S/S_{22})? \tag{4.1}$$

As the Schur complement of an accretive-dissipative matrix is again accretive-dissipative [8, Property 6], $(T/T_{22})\sigma(S/S_{22})$ in the preceding question is well defined. Unfortunately, the alleged inequalities fail, as there are counterexamples from numerical experiments.

Example 4.3. *Firstly, consider*

$$T = \begin{bmatrix} 1.0243 & 0.1853 \\ 0.1853 & 3.5998 \end{bmatrix} + i \begin{bmatrix} 6.4574 & -2.2991 \\ -2.2991 & 2.7951 \end{bmatrix},$$
$$S = \begin{bmatrix} 2.0098 & -0.7586 \\ -0.7586 & 0.9167 \end{bmatrix} + i \begin{bmatrix} 4.5054 & 2.1678 \\ 2.1678 & 2.0539 \end{bmatrix},$$

one has $(T+S)/(T+S)_{22} = 2.9854 + 10.9817i$ and $(T/T_{22}) + (S/S_{22}) = 6.1415 + 9.3255i$. Thus (4.1) fails for $\sigma = +$. The difference between $+$ and ∇ is a factor 2.

Next, consider

$$T = \begin{bmatrix} 1.1430 & 0.2011 \\ 0.2011 & 2.2426 \end{bmatrix} + i \begin{bmatrix} 13.1814 & 9.6876 \\ 9.6876 & 7.8507 \end{bmatrix},$$
$$S = \begin{bmatrix} 5.2840 & 1.9396 \\ 1.9396 & 1.3959 \end{bmatrix} + i \begin{bmatrix} 4.6687 & 1.9980 \\ 1.9980 & 6.0727 \end{bmatrix},$$

numerical experiment shows the real part of $(T\sharp S)/(T\sharp S)_{22}$ is 2.2423 and the real part of $(T/T_{22})\sharp(S/S_{22})$ is 3.9582. Thus (4.1) fails for $\sigma = \sharp$.

Finally, consider

$$T = \begin{bmatrix} 1.3893 & 0.5787 \\ 0.5787 & 2.7774 \end{bmatrix} + i \begin{bmatrix} 3.1981 & -2.5932 \\ -2.5932 & 3.1951 \end{bmatrix},$$

$$S = \begin{bmatrix} 6.3055 & 1.7288 \\ 1.7288 & 1.2695 \end{bmatrix} + i \begin{bmatrix} 0.9966 & -0.3220 \\ -0.3220 & 1.6571 \end{bmatrix},$$

numerical experiment shows that $(T!S)/(T!S)_{22} = 2.7445+1.6561i$ *and* $(T/T_{22})!(S/S_{22}) = 3.7687 + 2.0181i$. *Thus (4.1) fails for* $\sigma =!$.

Part of the reason for the invalidity of (4.1) lies in the fact that "operator reverse monotonicity of the inverse" does not hold for accretive-dissipative matrices. Below are explicit examples.

Example 4.4. *Let* $T = I + 2iI$, $S = I + iI$, *so* $T \geq S$. *Simple calculation gives* $T^{-1} = \frac{1}{5}I - \frac{2}{5}iI$ *and* $S^{-1} = \frac{1}{2}I - \frac{1}{2}iI$, *so there is no ordering between* T^{-1} *and* S^{-1}. *As an another example, let* $T = 2I + iI$, $S = \frac{1}{3}I + iI$, *so* $T \geq S$. *Simple calculation gives* $T^{-1} = \frac{2}{5}I - \frac{1}{5}iI$ *and* $S^{-1} = \frac{3}{10}I - \frac{9}{10}iI$. *This shows it is possible that* $T^{-1} \geq S^{-1}$.

However, with the aid of (3.6) and (4.1), one can still finds the lower bound of $T\sigma S$ in terms of their real and imaginary parts: let $T, S \in \mathbb{M}_n^{++}$ as in (1.2) and be comformably partitioned, then

$$\begin{aligned} (T\sigma S)/(T\sigma S)_{22} &= (A\sigma C + iB\sigma D)/(A\sigma C + iB\sigma D)_{22} \\ &\geq (A\sigma C)/(A\sigma C)_{22} + i(B\sigma D)/(B\sigma D)_{22} \\ &\geq (A/A_{22})\sigma(C/C_{22}) + i(B/B_{22})\sigma(D/D_{22}). \end{aligned}$$

For the parallel sum, we have an equality.

Proposition 4.5. *Let* $T, S \in \mathbb{M}_n^{++}$ *be partitioned as in (4.1), then*

$$(T : S)/(T : S)_{22} = (T/T_{22}) : (S/S_{22}). \tag{4.2}$$

Proof. For any nonsingular $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ with X_{22} being nonsingular, we know (see e.g. [10, p. 18])

$$X^{-1} = \begin{bmatrix} (X/X_{22})^{-1} & \star \\ \star & \star \end{bmatrix}, \tag{4.3}$$

where $\star$s are blocks irrelevant to our discussion. Observe that

$$((T : S)^{-1})_{11} = (T^{-1} + S^{-1})_{11} = (T^{-1})_{11} + (S^{-1})_{11}.$$

Using (4.3), the above equality is the same as

$$((T : S)/(T : S)_{22})^{-1} = (T/T_{22})^{-1} + (S/S_{22})^{-1}.$$

Taking inverse, we have (4.2). □

Remark 4.6. Indeed, Proposition 4.5 holds for any two matrices provided all the relevant inverses exist. Also, we remark that the inequality (3.16) in [14, Corollary 3.6] should be equality.

4.2 A few final words

This paper defines the arithmetic, geometric and harmonic mean of two accretive-dissipative matrices. It may serve as a basis to study the Loewner order of complex matrices, in particular the Loewner order of accretive-dissipative matrices. We have seen though not all properties in Hermitian positive definite matrices have its (direct) counterparts in the cone of accretive-dissipative matrices, some connection and analogy still exist. It is expected that more interesting results on this aspect can be found in the near future.

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Minghua Lin
Department of Applied Mathematics,
University of Waterloo,
Waterloo, ON, N2L 3G1, Canada.
mlin87@ymail.com